Solution to Class Exercise 6

1. Evaluate
$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2}\, dxdy \ .$$

   Solution. Let $t = 2x - y \in [0, 2]$ and $y = y$. Then
$$\frac{\partial(t, y)}{\partial(x, y)} = 2 \ .$$
   Therefore, the integral is equal to
$$\begin{aligned}\int_0^2 \int_0^4 y^3 t e^{t^2} \frac{1}{2}\, dtdy &= \frac{1}{2}\Big(\int_0^2 y^3\, dy\Big)\Big(\int_0^4 t e^{t^2}\, dy\Big) \\ &= \frac{1}{2}\Big[\frac{y^4}{4}\Big]_0^2 \Big[\frac{e^{t^2}}{2}\Big]_0^4 \\ &= \frac{1}{2} \cdot 4 \cdot \frac{e^{16}-1}{2} \\ &= e^{16} - 1\end{aligned}$$

2. Evaluate
$$\iint_D \frac{y}{x}\, dA$$
   where D is the region bounded by $y = x/2, y = 0, x^2 - y^2 = 4, x^2 - y^2 = 1$.

   Solution. First, notice that $D = D_+ \cup D_-$ is a disjoint union of two subregions D_+ and D_-, where
$$D_+ = D \cap \{(x, y) \in \mathbb{R}^2 | x, y > 0\}$$
$$D_- = D \cap \{(x, y) \in \mathbb{R}^2 | x, y < 0\}$$
   Therefore, by Theorem 1.8 of Chapter 1 of the lecture note,
$$\iint_D \frac{y}{x}\, dA = \iint_{D_+} \frac{y}{x}\, dA + \iint_{D_-} \frac{y}{x}\, dA$$
   Next, observe that the integrand $f(x, y) = y/x$ is invariant under the substitution $\Phi(x, y) := (-x, -y)$, i.e. $f(x, y) = f(-x, -y) = f(\Phi(x, y))$. Also, $\Phi : D_+ \to D_-$ is a C^1-diffeomorphism with $J_\Phi = 1$. Therefore, by the Change of Variables Formula,
$$\iint_{D_+} \frac{y}{x}\, dA = \iint_{D_-} \frac{y}{x}\, dA$$
   Therefore, we have
$$\iint_D \frac{y}{x}\, dA = \iint_{D_+} \frac{y}{x}\, dA + \iint_{D_-} \frac{y}{x}\, dA = 2\iint_{D_+} \frac{y}{x}\, dA$$

Introduce new variables $u = y/x \in [0, 1/2]$ and $v = x^2 - y^2 \in [1, 4]$. This defines a C^1-diffeomorphism from D_+ to $D_1 := [0, 1/2] \times [1, 4]$ with

$$\frac{\partial(u, v)}{\partial(x, y)} = -2 + 2x^2/y^2 = 2(u^2 - 1) \ ,$$

which is negative for $u \in [0, 1/2]$. Therefore,

$$\begin{aligned}
\iint_D \frac{y}{x}\, dA &= 2\iint_{D_+} \frac{y}{x}\, dA \\
&= 2\iint_{D_1} u \times \left|\frac{1}{2(u^2-1)}\right| dA(u, v) \\
&= 2\int_0^{1/2}\int_1^4 \frac{u}{2(1-u^2)}\, dvdu \\
&= -\frac{3}{2}\log\frac{3}{4} \ .
\end{aligned}$$